Chapter 5

Conceptual Basis of General Relativity

5.1 The story so far

We are at last ready to embark on our central task, namely, that of extending Special Relativity to a theory which incorporates gravitation. In this section we will consider the physical principles which guided Einstein in his search for the general theory. There is a school of thought that considers this an unnecessary process, but rather argues that it is sufficient to first state the theory and then investigate its consequences. There seems little doubt, however, that consideration of these physical principles helps gives insight into the theory and promotes understanding. The mere fact that they were important to Einstein would seem sufficient to justify their inclusion. If nothing else, it will help us understand how one of the greatest achievements of the human mind came about.

Many physical theories today start by specifying a Lagrangian from which everything flows and we could adopt the same attitude with General Relativity. Although this is a very beautiful way about going about things, in taking that approach we would miss out on gaining some understanding of how the framework of General Relativity is different from that of Newtonian theory and Special Relativity. Moreover if we discover limitations in the theory, then there is more chance of rescuing it by investigating the physical basis of the theory rather than simply tinkering with the mathematics - an unfortunate trait of much of modern theoretical physics these days!

But before we embark on this exciting journey of discovery, we must first remind ourselves of where we have got to. So far we have only discussed Special Relativity. Here forces have only played a background role and we have never introduced gravitation explicitly as a possible force. One aspect of Special Relativity is the existence of a global inertial frame, all of whose coordinate points are always at rest relative to the origin, and all of whose clocks run at the same rate relative to the origin's clock. From Einstein's postulates we were led to the idea of the spacetime interval ds^2 which gave an invariant geometrical meaning to certain physical statements. We discovered that the mathematical function that calculates the spacetime interval is the metric, and so the metric of Special Relativity is defined physically by lengths of rods and the readings of clocks. This closeness to experiment is of course its strength.

Let us now ask the following question:

- **Is is possible to construct a frame in which all clocks run at the same rate?**

This is a crucial question and we will show that in a non - uniform gravitational field the answer, experimentally , is **NO**.

- **So: Gravitational fields are incompatible with global Special Relativity i.e. it is impossible to construct a global inertial frame.**

We shall see that in small regions of spacetime [regions small enough that non - uniformities of the gravitational field are too small to measure] one can always construct a "Local Inertial Frame" [LIF]. In this sense we will have to build Special Relativity into a more general theory.

5.2 The gravitational redshift experiment

This "gedänken" experiment was first suggested by Einstein. Suppose we have a tower of height h on the surface of the earth. A particle of rest mass m is dropped from the top of the tower and falls freely with the acceleration g [see Figure 5.1].

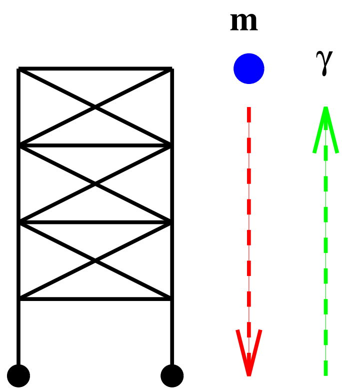

Figure 5.1: The Gravitational redshift experiment.

It reaches the ground with a velocity $v = \sqrt{2gh}$, so its total energy E, as measured by an observer at the foot of the tower is

$$E = mc^2 + \frac{1}{2}mv^2 + O(v^4) = mc^2 + mgh + O(v^4) \ . \tag{5.1}$$

Suppose the observer has some magical method of converting all this energy into a photon of the same energy [this is a thought experiment after all!]. Upon its arrival at the top of the tower with energy $\bar{E}$ the photon is again magically changed into a particle of rest mass $\bar{m} = \bar{E}/c^2$. It must be that $\bar{m} \leq m$; otherwise, perpetual motion could result, so $\bar{E} = mc^2$. We therefore obtain:

$$\frac{\bar{E}}{E} = \frac{mc^2}{mc^2 + mgh + O(v^4)} = 1 - \frac{gh}{c^2} + O(v^4) \ , \tag{5.2}$$

and since $E = h\nu$ and $\bar{E} = h\bar{\nu}$ we find:

$$\bar{\nu} = \nu \left(1 - \frac{gh}{c^2} \right) \ . \tag{5.3}$$

We therefore predict that a photon climbing in the earths gravitational field will lose energy and will consequently be redshifted. the redshift is:

$$Z_g = \frac{\nu - \bar{\nu}}{\nu} = \frac{gh}{c^2} \ . \tag{5.4}$$

This was tested by Pound and Snider in 1965 using the Mossbauer effect [photons from atomic decay peak sharply at a particular frequency]. They measured the redshift experienced by a 14.4 Kev γ rays from the decay of ^{57}Fe in climbing up

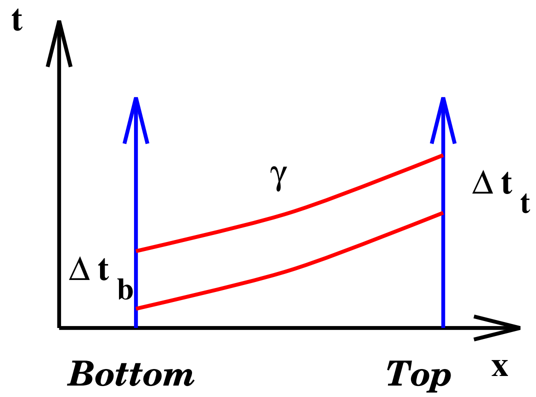

Figure 5.2: Minkowski geometry for the Pound - Snider Experiment.

a 20 m tower by determining the speed at which a detector at the top must be moved in order to maximize the detection rate i.e. the velocity blueshift balances the gravitational redshift. They found:

$$Z_g = 2.57 \pm 0.26 \times 10^{-15} \; . \tag{5.5}$$

This experimental verification of Einstein's thought experiment is a death - blow of one's chances of finding a simple special relativistic theory of gravity!

5.3 Non - existence of an inertial frame at rest on earth

If Special Relativity is to be valid in a gravitational field, it is a natural first guess to assume that the "laboratory" frame at rest on earth is an inertial frame. Let us draw a spacetime diagram representing the above experiment [see Figure 5.2]. We consider light as a wave, and look at two successive crests of the wave as they move upward in the gravitational field. The top and bottom of the tower have vertical world lines in this diagram since they are at rest. Light is shown moving on a curved line, to allow for the possibility that gravity may act on on light in an unknown way, deflecting it from a null path. But no matter how light is affected by gravity the effect must be the same on both wave crests, since the gravitational field is not time dependent. Therefore the two crests ' paths are congruent, and we conclude from this hypothetical Minkowski geometry that

$$\Delta t_t = \Delta t_b \ . \tag{5.6}$$

But we know that $\Delta t = \frac{1}{\nu}$, and since the Pound - Snider experiment tells us that $\nu_b > \nu_T$ we know that $\Delta t_t > \Delta_b$. Therefore we have to conclude that our answer using Minkowski geometry is wrong!

- **So the reference frame at rest on earth is not inertial!!**

Is this the end of Special Relativity!!?..... Not quite. We have shown that a particular frame is not inertial, not that there are no inertial frames. We will find that there are certain frames which are inertial in a restricted sense, but before we define these frames let us first consider what we mean by the mass of a body.

5.4 Mass in Newtonian theory

So far we have been rather vague about what we mean by the mass of a body. Even in Newtonian theory we can ascribe three masses to any body which describe quite different properties:

- **Inertial mass m^I: This is a measure of its resistance to change in motion or inertia.**

- **Passive gravitational mass m^P: This is a measure of its reaction to a gravitational field.**

- **Active gravitational mass m^A: This is a measure of its source strength for producing a gravitational field.**

Let us discuss each of these in turn. Inertial mass m^I is the quantity occurring in Newton's second law [$\mathbf{F} = m^I \mathbf{a}$]. It is a measure of a body's inertia. Note that as far as Newtonian theory, this mass has nothing to do with gravitation. The other two masses however do.

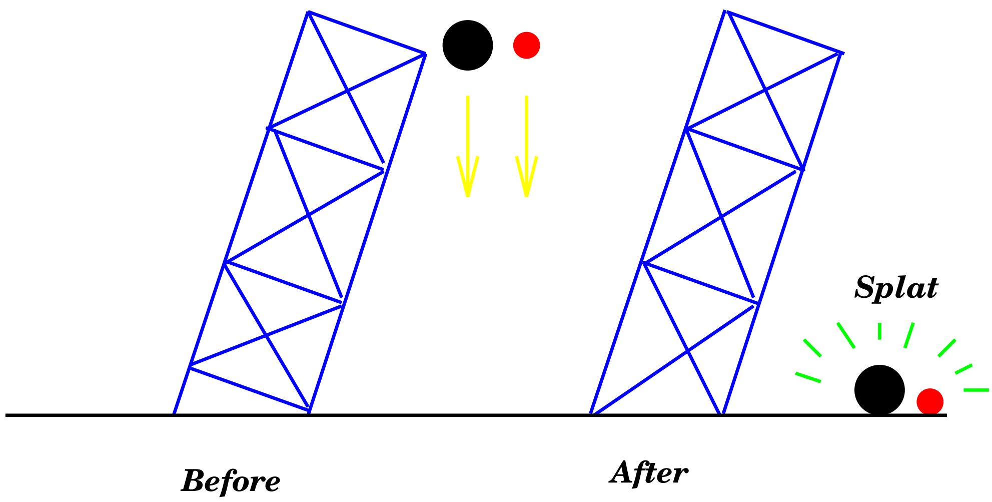

Figure 5.3: The Galileo Piza experiment.

Passive gravitational mass m^P measures a body's response to being placed in a gravitational field. Let the gravitational potential at some point be Φ, then if m^P is placed at this point, it will experience a force on it given by

$$\mathbf{F} = -m^P \nabla \Phi \ . \tag{5.7}$$

On the other hand active gravitational mass m^A measures the strength of the gravitational field produced by the body itself. If m^A is placed at the origin, then the gravitational potential at any point a distance r from the origin is given by

$$\Phi = -\frac{Gm^A}{r} \ . \tag{5.8}$$

We will now see how these three masses are related in the Newtonian framework.

Galileo discovered in his famous Pisa experiments [see Figure 5.3] that when two bodies are dropped from the same height, they reach the ground together irrespective of their internal composition.

Let's assume that two particles of inertial mass m_1^I and m_2^I and passive gravitational mass m_1^P and m_2^P are dropped from the same height in a gravitational field. We have:

$$\begin{aligned} m_1^I \mathbf{a}_1 &= \mathbf{F}_1 = -m_1^P \nabla \Phi \ , \\ m_2^I \mathbf{a}_2 &= \mathbf{F}_2 = -m_2^P \nabla \Phi \ . \end{aligned} \tag{5.9}$$

The observational result is $a_1 = a_2$ from which we get on dividing

$$\frac{m_1^I}{m_1^P} = \frac{m_2^I}{m_2^P} \ . \tag{5.10}$$

Repeating this experiment with other bodies, we see that this ratio is equal to a universal constant α say. By a suitable choice of units we can take $\alpha = 1$, from which we obtain the result:

- **inertial mass = passive gravitational mass.**

This equality is one of the best tested results in physics and has been verified to 1 part in 10^{12}.

In order to relate passive gravitational mass to active gravitational mass, we make use of the observation that nothing can be shielded from a gravitational field. Consider two isolated bodies situated at points Q and R moving under their mutual gravitational attraction. The gravitational potential due to each body is

$$\Phi_1 = -\frac{Gm_1^A}{r} \ , \qquad \Phi_2 = -\frac{Gm_2^A}{r} \ . \tag{5.11}$$

The force which each body experiences is

$$\mathbf{F}_1 = -m_1^P \nabla_Q \Phi_2 \qquad \mathbf{F}_2 = -m_2^P \nabla_R \Phi_1 \ . \tag{5.12}$$

If we taken the origin to be Q then the gradient operators are

$$\nabla_R = \hat{\mathbf{r}} \frac{\partial}{\partial r} = -\nabla_Q \ , \tag{5.13}$$

so that

$$\mathbf{F}_1 = \frac{Gm_1^P m_2^A}{r^2} \hat{\mathbf{r}} \ , \qquad \mathbf{F}_2 = \frac{Gm_2^P m_1^A}{r^2} \hat{\mathbf{r}} \ . \tag{5.14}$$

But by Newton's third law $\mathbf{F}_1 = \mathbf{F}_2$, and so we conclude that

$$\frac{m_1^P}{m_1^A} = \frac{m_2^P}{m_2^A} \ , \tag{5.15}$$

and using the same argument as before, we see that

- **Passive gravitational mass = active gravitational mass.**

That is why in Newtonian theory we can simply refer to the mass m of a body where

$$m = m^I = m^P = m^A \ . \tag{5.16}$$

This may see obvious to you, but it has very deep significance and Einstein used it as the central pillar for his equivalence principle.

5.5 The principle of equivalence

One important property of an inertial frame is that a particle stays at rest or moves with uniform velocity unless it is acted on by a force [Statement of Newton's first law]. In the last section we discovered that bodies freely falling in a gravitational field all accelerate at the same rate regardless of their internal composition. It follows that relative to a non - rotating freely falling frame, at least locally, particles remain at rest or move in straight lines with uniform velocity, since this frame accelerates at the same rate as particles do. By locally we mean that observations are confined to a region over which the variation of the gravitational field is un - observably small. This leads to the following statement of the principle of equivalence.

- **POE 1: There are no local experiments which can distinguish non - rotating free fall in a gravitational field from uniform motion in space in the absence of a gravitational field.**

We conclude that a non - rotating freely falling frame is a local inertial frame. Let's check this by viewing the Pound - Snider experiment from the view point of a freely falling frame.

Let us take the particular frame to be at rest when the photon begins its upward journey and falls freely after that. Since the photon rises a distance h, it takes a time $\Delta t = \frac{h}{c}$ to arrive at the top. In this time, the frame has acquired a velocity $v = g\Delta t = \frac{gh}{c}$ downward relative to the tower.

We can now use the redshift formula from section 2.2 to calculate the photon's frequency relative to the freely falling frame:

$$\frac{\nu_{ff}}{\nu_t} = \sqrt{\frac{1 + gh/c^2}{1 - gh/c^2}} \approx 1 + \frac{gh}{c^2} \,, \tag{5.17}$$

so

$$\begin{aligned} \nu_{ff} &= \nu_t \left(1 + \frac{gh}{c^2}\right) \\ &\approx \nu \left(1 + \frac{gh}{c^2}\right)\left(1 - \frac{gh}{c^2}\right) \approx \nu \,, \end{aligned} \tag{5.18}$$

since we neglect terms of higher order [as we did in section 5.2]. So there is no redshift in a freely falling frame thus confirming that it is a local inertial frame.

Einstein noted one other coincidence in Newtonian theory which proved to be of great importance in formulating the principle of equivalence. All inertial forces are proportional to the mass of the body experiencing the force. There is one other force which behaves in the same way, that is the force of gravitation. For, if we drop two bodies in the earths gravitational field, they experience forces $m_1 g$ and $m_2 g$ respectively. This coincidence suggested to Einstein that the two effects should be considered as arising from the same origin. Thus he suggested that we treat gravitation as as inertial effect as well, in other words as an effect which arises from not using an inertial frame. Comparing the force mg of a falling body with the inertial force ma suggests the following version of the principle of equivalence.

- **POE 2: A frame linearly accelerated relative to an inertial frame in special relativity is locally identical to a frame at rest in a gravitational field.**

These two versions of the principle of equivalence [POE 1, POE 2] can be vividly clarified by considering the famous "gedänken" experiments of Einstein called the lift experiments.

We consider an observer confined in a lift with no windows in it or other methods of communication with the outside world. The observer is allowed equipment to carry out simple dynamical experiments. The object of the exercise is to try and

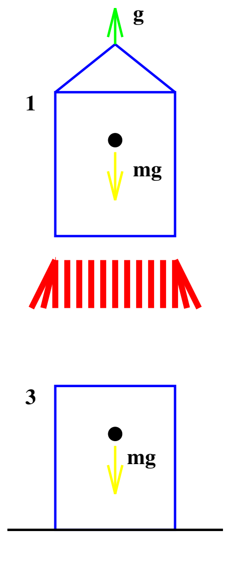

Figure 5.4: The lift experiments

determine the observers state of motion. Let us consider four cases [see Figure 5.4].

- **Case 1: The lift is placed in a rocket ship in a part of the universe far removed from gravitating bodies. The rocket is accelerated forward with a constant acceleration g relative to an inertial observer. The observer releases a body from rest and sees it fall to the floor with acceleration g.**

- **Case 2: The rocket motor is switched off so that the lift undergoes uniform motion relative to the inertial observer. A released body is found to remain at rest relative to the observer in the lift.**

- **Case 3: The lift is next placed on the surface of the earth, whose rotational and orbital motions are ignored. A released body is found to fall to the floor with acceleration g.**

- **Case 4: Finally, the lift is placed in an evacuated lift shaft and allowed to fall freely towards the center of the earth. A released body is found to remain at rest relative to the observer.**

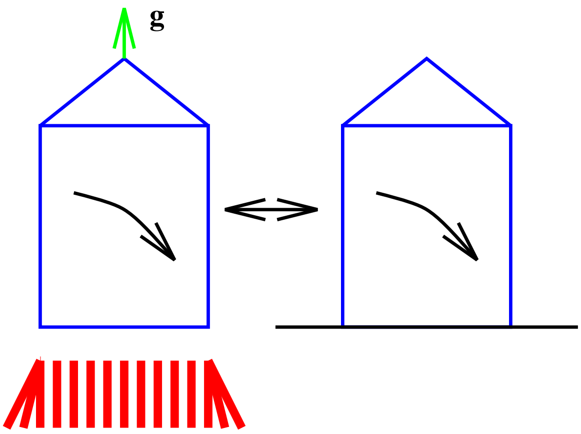

Figure 5.5: The bending of light

Clearly, from the point of view of the observer in the lift, cases 1 and 3 are indistinguishable, as required by POE 2, and cases 2 and 4 are indistinguishable, as required by POE 1.

This is the principle of equivalence between gravity and acceleration, and is a corner stone of the theory of General Relativity. In more modern terminology, what we have described is called the weak equivalence principle, "weak" because it refers only to gravity. We shall later use the strong equivalence principle, which says that one can discover how all the laws of physics behave in a gravitational field by postulating that their laws in a freely falling inertial frame are identical to their laws in Special Relativity i.e. when there are no gravitational fields.

5.6 The principle of equivalence in action

Let us now look at two examples of how we use the principle of equivalence.

5.6.1 Effect of gravity on light

Since we can think of light as having a mass $\frac{h\nu}{c^2}$, relative to an observer in an accelerated lift, light will curve downwards [see Figure 5.5]. Therefore by the principle of equivalence, light will also be bent in a gravitational field [examples: light deflection due to the sun's gravitational field; observed during a total solar eclipse. Lensing of cosmological sources.].

5.6.2 Effect of gravity on time

Consider a rocket of height h undergoing acceleration g relative to an outside observer. Let a light ray be emitted from the top (B) at time $t = 0$ and be received at the bottom (A) at time $t = t_0$ in the frame of the outside observer [see Figure 5.6]. A second ray is emitted at $t = \Delta\tau$ and received at $t = t_0 + \Delta t$.

One can show [EXERCISE 5.1] that

$$\Delta\tau = \Delta t\left(1 + \frac{gh}{c^2}\right) , \tag{5.19}$$

where we have assumed that $\frac{gh}{c^2} \ll 1$ (i.e non-relativistic motion).

Using the equivalence principle we know that the same relation must apply of there is a difference in gravitational potential $\Phi_B - \Phi_A$ between two points B and A in a gravitational field. i.e.

$$\Delta\tau = \Delta t\left(1 + \frac{\Phi_B - \Phi_A}{c^2}\right) . \tag{5.20}$$

If $A \to \infty$ where $\Phi = 0$ [no gravitational field] and B is taken to be a general point with position vector $\mathbf{r}$, we expect

$$\Delta\tau = \Delta t\left(1 + \frac{\Phi(\mathbf{r})}{c^2}\right) . \tag{5.21}$$

Since $\Phi(\mathbf{r})$ is negative, the time measured on $B's$ clock [as seen by A at infinity] is less than the time measured on $A's$ clock, i.e. clocks run slow in a gravitational field.

One can interpret this by imagining that the spacetime metric has the non-Minkowski form:

$$ds^2 = -\left(1 + \frac{2\Phi}{c^2}\right)c^2dt^2 + dx^2 + dy^2 + dz^2 . \tag{5.22}$$

Then the proper time measured by a clock at fixed (x, y, z) in a time Δt measured at infinity is

$$d\tau = \frac{1}{c}\sqrt{-ds^2} , \tag{5.23}$$

therefore

$$\Delta\tau = \left(1 + \frac{\Phi}{c^2}\right)\Delta t , \tag{5.24}$$

for $\frac{\Phi}{c^2} \ll 1$. This corresponds to spacetime curvature.

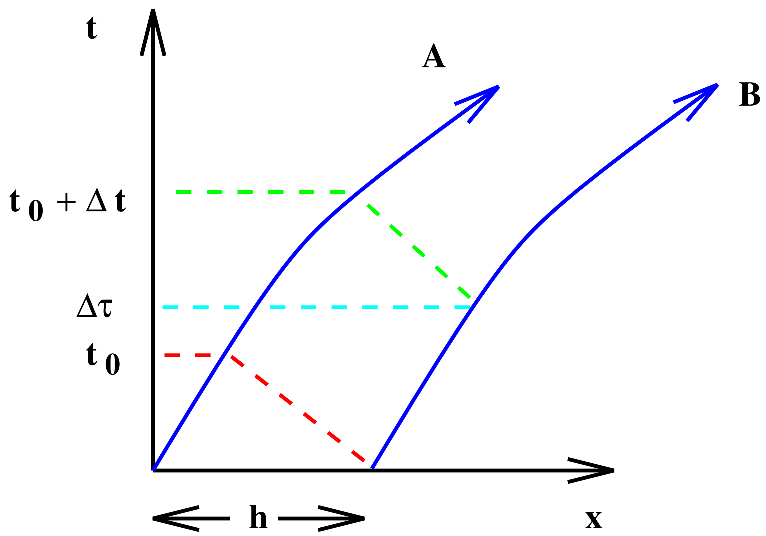

Figure 5.6: Space time diagram of rocket undergoing uniform acceleration g

5.6.3 Towards spacetime curvature

The world lines of free particles have been our probe of the possibility of constructing inertial frames. In Special Relativity two such world lines which begin parallel to each other remain parallel, no matter how far extended. This is exactly the property that straight lines have in Euclidian geometry. It is natural, therefore, to discuss the geometry of spacetime as defined by the worldlines of free particles. In these terms Minkowski space is a flat space [it is not Euclidian space because the metric is indefinite $(-+++)$]. To discuss non - uniform gravitational fields let us consider the lift experiments again, this time making them big enough so that variations of the gravitational field can be measured [see Figure 5.7].

- **Case 1: From the point of view of the observer in the lift, the two bodies fall to the ground parallel to each other.**
- **Case 2: The bodies remain at rest relative to the observer.**
- **Case 3: The two bodies fall towards the center of the earth and hence fall on paths which converge.**

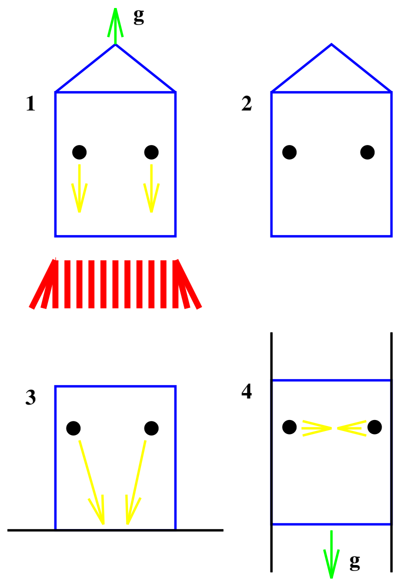

Figure 5.7: The lift experiments again

- **Case 4: The bodies appear to the observer to move closer together because they are falling on lines that converge towards the center of the earth.**

We thus conclude that gravitational spacetime is therefore not flat - it is curved. A classic example, which we will us many times to illustrate our ideas, is the surface of a sphere or balloon. Locally straight lines on a sphere extend to great circles and two great circles always intersect [at the poles]. Nevertheless, close to any point, we can pretend the geometry is flat. This is true also for Riemannian spaces: they all are locally flat, but the locally straight lines [geodesics] do not usually remain parallel.

Einstein's important advance was to see the similarity between Riemannian spaces and gravitational physics. He identified the trajectories of freely falling particles with the geodesics of a curved geometry: They are locally straight since spacetime admits local inertial frames, but globally they do not remain parallel.